

Complex Dynamics of A Class of Nonlinear Financial Systems and Financial Security

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Abstract

During the operation of the financial system, financial chaos such as the severe turbulence of the financial market and the financial crisis occurred due to deterministic instability, which brought a great negative impact on economic growth and financial risk. The dynamics of a financial chaotic system is analyzed, the central manifold function is obtained by using the central manifold theorem. When the parameters of the system take different ranges of values, the dynamic properties of the system change from simple to complex, at last the system exhibits chaotic dynamic properties. The numerical simulation is consistent with the theoretical analysis.

Keywords

Financial model; Financial security; Central manifold theorem; Chaos.

1. INTRODUCTION

Chaos[1,2] is a unique and widely existing aperiodic motion form of nonlinear system. Chaos is widely used in natural science and social science. The early research on chaotic motion can be traced back to the simulation of atmospheric turbulence between two infinite planes by Lorenz in 1963[3]. Subsequently, Hénon and Rossler studied a two-dimensional system and a three-Dimensional system respectively[4,5], they obtained randomness result of solution. Ruelle, May and Feigenbaum have further studied the characteristics of this kind of random motion[6-8], thus creating a new research direction of chaos. Later investigations found that there exist chaotic attractors in the chemical reaction model[9] and electrical circuit[10].

The study of chaotic dynamics and chaos control have attracted extensive attention in biology. It has more complex dynamic properties and applications in high-dimensional biological systems [11,12]. In recent years, a further mathematical study on the reports of existence of multi-scroll chaotic attractors[13,14]. And a 4D chaotic system and its multistability were studied in[15]. With the diversity of mathematical research methods, more and more complex systems are studied in detail. Such chaotic systems display more complex dynamical behaviors and have applications in the fields of engineering and economy. Chaos in economics has had a great impact on the western economists since it was first discovered in 1985, because the emergence of chaos in the economic systems means that the macroeconomic movement has internal unstability and complexity[16].

In the fields of finance, stock and social economy, due to the interaction of nonlinear factors, various economic problems are becoming more and more complex, and the evolution process has been found from low dimension to high dimension. The internal structure of the system also presents diversity and complexity[17,18]. This kind of system has extremely complex phenomena and external characteristics

Therefore, it becomes more and more important to systematically and deeply study the internal structural characteristics of this kind of complex economic system, and reveal the causes of complex phenomena by studying the instability, bifurcation, periods doubling bifurcation, the position of each bifurcation point and the path of chaos. Zhang et al. studied a 4D chaos financial system[19]. It provides a theoretical basis and practical operation method for the analysis, prediction and control of complex continuous economic system. Synchronization of the model were considered in their work.

The complexities of the financial systems were studied and a series of control strategies to chaotic financial systems were proposed[20,21]. As interaction of elements of these financial systems, they can exhibit complex dynamic behaviors. It is particularly important to reveal the development law of the financial system and control the chaotic state of the system. Many scholars have studied some nonlinear financial models.

Huang[16] proposed the classical financial system (1), and Ma[22,23] gave its complex dynamics.

$$\begin{cases} \dot{x} = z + (y - a)x, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz. \end{cases} \quad (1)$$

Where x represents interest rate, y represents investment demand, z represents price index, a represents savings, b represents unit investment cost, and c represents commodity demand elasticity.

Wang[24] established new financial systems, analyzed the dynamics of the financial systems and studied the control methods, so that people have a deeper understanding of the financial systems. It is found that the factors affecting the change of interest rate are not only investment demand and price index, but also related to the average profit margin, and the average profit margin is positively proportional to the interest rate. Previous research and methods mainly focused on numerical calculation and numerical simulation without detailed theoretical derivation as support.

In this work, the financial system (1) is studied by using central manifold theory and verified numerically. The conditions for the local complex behavior of this kind of financial system were given. This research will help people understand the leverage of various financial policies. We construct a mathematical model to study a financial system using differential equations. The model captures the interaction between a variety of financial factors. This paper is organized as follows. In Section 2, we formulate the model. In Section 3, we perform numerical simulations to study the complexity of the model. Conclusions are drawn in Section 4.

2. FINANCIAL SYSTEM MODEL AND THE BIFURCATION AND STABILITY ANALYSIS AT EQUILIBRIUM POINTS.

Considering the positive relationship between average profit margin and interest rate, a control term kx is added to the general model, we obtain the following improved financial system model

$$\begin{cases} \dot{x} = z + (y - a)x + kx, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz. \end{cases} \quad (2)$$

Where x represents interest rate, y represents investment demand, z represents price index, a represents savings, b represents unit investment cost, and c represents commodity demand elasticity.

k is the control parameter.

In[25], the Lyapunov exponent under different parameters of the system (2) is calculated. The analysis shows that the selection of the parameters is very important for controlling financial chaos system.

Let

$$\begin{cases} z + (y - a)x + kx = 0, \\ 1 - by - x^2 = 0, \\ -x - cz = 0. \end{cases} \tag{3}$$

When $c - b - abc + bck < 0$, the system has a unique equilibrium point $E_1 = (0, 1/b, 0)$. When $c - b - abc + bck > 0$, the system has three equilibrium points E_1, E_2, E_3 ,

$$\begin{aligned} E_1 &= (0, 1/b, 0), \\ E_2 &= \left(-\frac{\sqrt{-b + c - abc + bck}}{\sqrt{c}}, \frac{1 + ac - ck}{c}, \frac{\sqrt{-b + c - abc + bck}}{c^{3/2}} \right), \\ E_3 &= \left(\frac{\sqrt{-b + c - abc + bck}}{\sqrt{c}}, \frac{1 + ac - ck}{c}, -\frac{\sqrt{-b + c - abc + bck}}{c^{3/2}} \right). \end{aligned} \tag{4}$$

We mainly research the topology and stability of the system (2) at the equilibrium point $E_1 = (0, 1/b, 0)$ when $c - b - abc + bck < 0$. Let $X = x, Y = y - 1/b, Z = z$, then system (1) becomes

$$\begin{cases} \dot{X} = (1/b - a)X + Z + XY, \\ \dot{Y} = -bY - X^2, \\ \dot{Z} = -X - cZ. \end{cases} \tag{5}$$

The equilibrium point of the system (1) becomes $(0, 0, 0)$. The characteristic equation of the linearized system corresponding to Equation (5) is

$$F(\lambda) = (\lambda + b)(\lambda^2 + (c + a - 1/b)\lambda + 1 + ac - c/b) = 0. \tag{6}$$

Then the three eigenvalues corresponding to Equation (6) are $\lambda_1 = -b, \lambda_2, \lambda_3$. The values of λ_2, λ_3 can be obtained from Equation (7),

$$\lambda^2 + (c + a - 1/b)\lambda + 1 + ac - c/b = 0. \tag{7}$$

As $-b + c - abc + bck < 0, \frac{b - c + abc - bck}{b} > 0$, the eigenvalues corresponding to the Equation (7) are discussed in three cases.

1). When $-b + c - abc + bck < 0, c + a - k - 1/b > 0$, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the equation $F(\lambda) = 0$ are all less than 0, the Equation (2) has a unique equilibrium point E_1 , and the equilibrium point E_1 is a stable sink.

2). When $-b + c - abc + bck < 0, c + a - k - 1/b < 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_1 = -b < 0$, E_1 is a saddle point.

3). When $-b + c - abc + bck < 0, c + a - k - 1/b = 0, \lambda_2, \lambda_3$ are a pair of pure imaginary roots and $\lambda_1 = -b < 0$. We take $\alpha = -(c + a - k - 1/b)$, then $\frac{\partial \alpha}{\partial a} \Big|_{a=a_0} = -1 \neq 0$ satisfies the cross-section condition of Hopf bifurcation. It means that when a is equal to a_0 , the trajectory of the system crosses the virtual axis, Hopf bifurcation occurs at the equilibrium point E_1 .

The three eigenvectors corresponding to the eigenvalues $\lambda_1 = -b, \lambda_2 = 0, \lambda_3 = (1 - c^2)/c$ are

$(0, 1, 0)^T, (1, 0, -1/c)^T$ and $(-1/c, 0, 1)^T$. The eigenvector of λ_1, λ_3 is expanded into a stable subspace E^s , and the eigenvector of λ_2 is expanded into a central subspace E^c .

Perform a linear transformation $T(u, v, w)^T$ on $(X, Y, Z)^T$,

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix}, T = \begin{pmatrix} 1 & -1/c & 0 \\ 0 & 0 & 1 \\ -1/c & 1 & 0 \end{pmatrix}, T^{-1} = \begin{pmatrix} c^2/(c^2 - 1) & 0 & c/(c^2 - 1) \\ c/(1 - c^2) & 0 & c^2/(1 - c^2) \\ 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

Bring Equation (8) into Equation (5)

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (c^2 - 1)/c & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} c^2(u - c^{-1}v)w/(c^2 - 1) \\ c^2(u - c^{-1}v)w/(1 - c^2) \\ -(u - c^{-1}v)^2 \end{pmatrix}. \tag{9}$$

The three eigenvalues of the new variable $(u, v, w)^T$ are still $\lambda_1 = -b, \lambda_2 = 0, \lambda_3 = (1 - c^2)/c$, but their corresponding eigenvectors become $(0, 0, 1)^T, (1, 0, 0)^T, (0, 1, 0)^T$ respectively. Therefore $E^s = \text{span}\{(u, v, w)^T\}, E^u = \emptyset$, the linear part of Equation (9) has been decoupled,

$$\dot{u} = \frac{c^2}{c^2 - 1} (u - c^{-1}v)w, \tag{10a}$$

$$\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} c^2 / (c^2 - 1) & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} \frac{c}{1 - c^2} (u - c^{-1}v)w \\ -(u - c^{-1}v)^2 \end{pmatrix}. \tag{10b}$$

The central manifold W^c is a tangent curve of the central subspace E^c . As W^c and E^c are tangent at the equilibrium point $(0, 0, 0)$, and the functions from u axis to v axis and w axis are respectively $h_1(u), h_2(u)$. The central manifold function expression is

$$W^c = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} = h(U) = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} \right\}, \tag{11}$$

$$h_1(u) = 0, h_2(u) = 0.$$

Where $h(U): R \rightarrow R^2$ is the differential homeomorphism on the neighborhood of the point $(0, 0, 0)$. Equation (10) is expressed in the form of power series as,

$$\begin{pmatrix} v \\ w \end{pmatrix} = h(U) = \begin{pmatrix} a_1u^2 + b_1u^3 + c_1u^4 + \dots \\ a_2u^2 + b_2u^3 + c_2u^4 + \dots \end{pmatrix}. \tag{12}$$

Bring Equation (12) into Equation (10b),

$$\begin{pmatrix} \dot{h}_1(u) \\ \dot{h}_2(u) \end{pmatrix} \begin{pmatrix} \frac{c^2}{1 - c^2} (u - c^{-1}v)h_2(u) \end{pmatrix} = \begin{pmatrix} h_1(u) + \frac{c^2}{1 - c^2} (u - c^{-1}h_1(u))h_2(u) \\ -bh_2(u) - (u - c^{-1}h_1(u))^2 \end{pmatrix}, \tag{13}$$

where $\dot{h}_i(u) = \partial h_i(u) / \partial u = 2a_iu + 3b_iu^2 + 4c_iu^3 + \dots$

Bring h, \dot{h} into Equation (13), after comparing the coefficients, the Equation v is obtained,

$$\begin{cases} O(u^2): \frac{c^2 - 1}{c} a_1 = 0, \\ O(u^3): \frac{c^2 - 1}{c} b_1 + \frac{c}{1 - c^2} a_2 = 0, \\ O(u^4): \frac{2a_1a_2c^2}{c^2 - 1} = \frac{c^2 - 1}{c} c_1 + \frac{c}{1 - c^2} \left(b_2 - \frac{a_1a_2}{c} \right). \end{cases} \tag{14a}$$

And the Equation w is obtained,

$$\begin{cases} O(u^2): \frac{c^2-1}{c}c_1 + \frac{c}{1-c^2}b_2 = 0, \\ O(u^3): \frac{2a_1a_2}{c} - b_1b_2 = 0, \\ O(u^4): \frac{2a_2^2c^2}{c^2-1} = \frac{2b_1}{c} - bc_2 - \frac{2a_1^2}{c^2}. \end{cases} \tag{14b}$$

$$a_1 = 0, b_1 = \frac{c^2}{b(c^2-1)^2}, c_1 = 0, a_2 = -\frac{1}{b}, b_2 = 0, c_3 = \frac{-2c}{b^3(c^2-1)^2}(b + c(c^2-1)). \tag{15}$$

The central manifold of the fourth power is

$$W^c = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} = \left\{ \begin{matrix} -\frac{c^2}{b(c^2-1)}u^3 + O(u^5) \\ -\frac{1}{b}u^2 - \frac{2c}{b^3(c^2-1)^2}(b + c(c^2-1)u^4) + O(u^5) \end{matrix} \right\} \right\}. \tag{16}$$

Bring h_1, h_2 into the Equation (10a),

$$\dot{u} = -\frac{c^2}{b(c^2-1)}u^3 - \frac{2c}{b^3(c^2-1)^3}(b + c(c^2-1)u^5) + O(u^6). \tag{17}$$

Since the coefficient of u^3 in Equation (17) is negative, the flow on the central manifold is asymptotically stable. The equilibrium point $(0, 0, 0)$ corresponding to the equilibrium point E_1 in Equation (2) is also asymptotically stable according the central manifold theorem.

According to the above analysis, it can be seen that under the condition of satisfying the case 3), when $c=1$, the system (1) bifurcates at the equilibrium point E_1 .

3. NUMERICAL SIMULATION

3.1. Impact of time range on the system

When we take the parameters $a=0.6, b=0.2, c=0.9$ and $k=0.5$, the initial values are $(0.02, 1, 0.04)$, considering the influence of the length of time range on the chaotic state of the system, we take the time $t \in (0, 11)$ and $t \in (0, 1000)$ respectively, and draw the phase diagrams.

1). When time $t \in (0, 15)$, the following phase diagram can be obtained (Figure. 1).

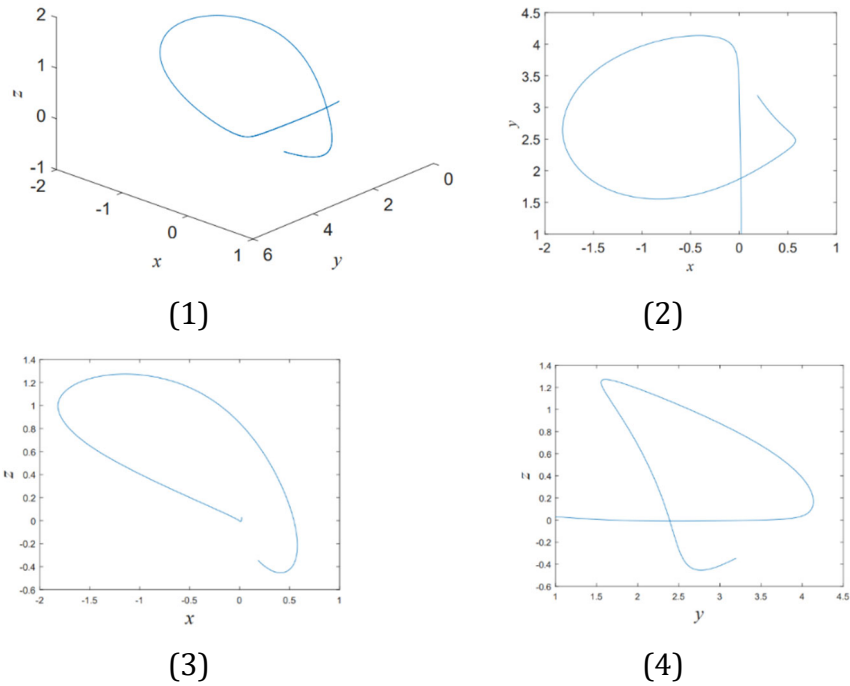


Figure 1. The phase diagram of Equation (2): (1) $(x-y-z)$, (2) $(x-y)$, (3) $(x-z)$, (4) $(y-z)$, in the range of $t \in (0, 15)$ ($a=0.6, b=0.2, c=0.9$ and $k=0.5$, the initial values $(0.02, 1, 0.04)$)

2). When time $t \in (0, 1500)$, the following phase diagram can be obtained (Figure. 2).

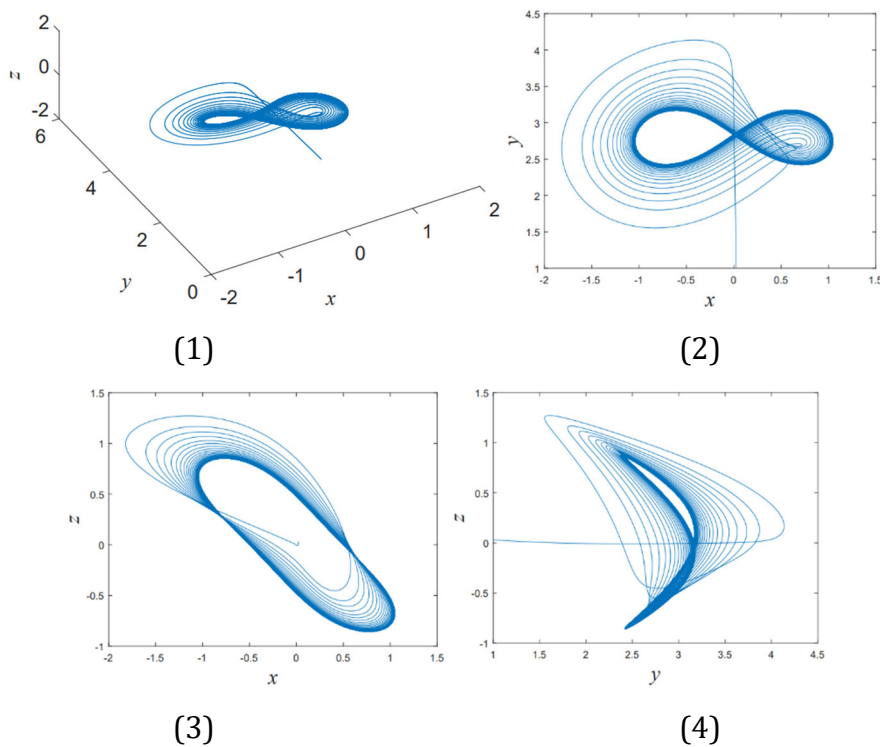


Figure 2. The phase diagram of Equation (2): (1) $(x-y-z)$, (2) $(x-y)$, (3) $(x-z)$, (4) $(y-z)$, in the range of $t \in (0, 1500)$ ($a=0.6, b=0.2, c=0.9$ and $k=0.5$, the initial values $(0.02, 1, 0.04)$)

3.2. Influence of parameters on the system.

Fixing the parameters $c=0.9$, $b=0.2$, $k=0.5$, we draw the bifurcation diagram of Equation (2) (Figure. 3). We find that when the parameter a is less than 1.46, the system is chaotic, and when the parameter a is greater than 1.46, the system exhibits a stable state. We draw the phase diagram of the system, taking the following parameters $a=0.8$ (Figure. 4), $a=1.2$ (Figure. 5), $a=1.8$ (Figure. 6). It can be observed that the system shows chaotic when the parameters: $a=0.8$, $a=1.2$ are taken. The phase diagrams are consistent with theoretical derivation and bifurcation diagrams. We also verified these conclusions by calculating the values of the Lyapunov exponent.

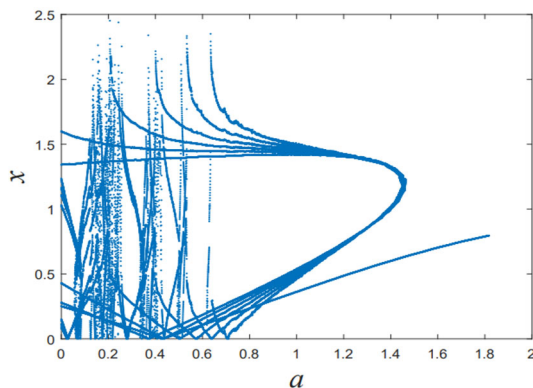


Figure 3. The bifurcation diagram (x - a) of Equation (2): ($c=0.9$, $b=0.2$, and $k=0.5$)

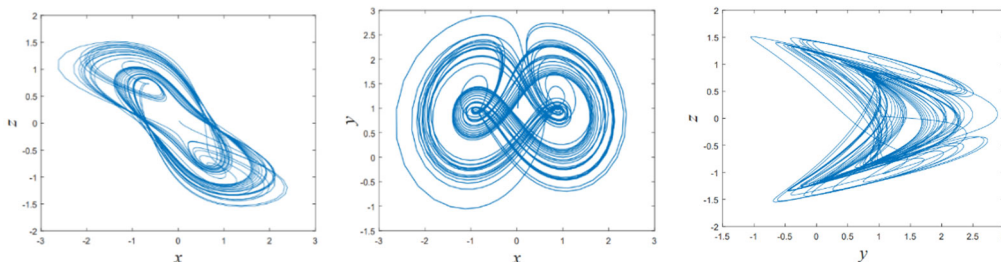


Figure 4. The phase diagram of Equation (2): (1) (x - y), (2) (x - z), (3) (y - z) as $a=0.8$ ($c=0.9$, $b=0.2$, and $k=0.5$)

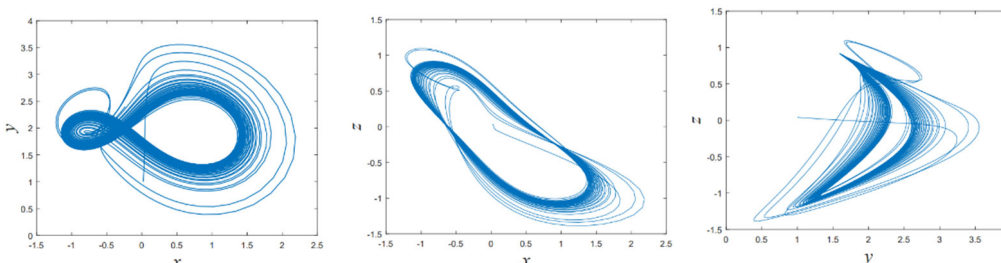


Figure 5. The phase diagram of Equation (2): (1) (x - y), (2) (x - z), (3) (y - z) as $a=1.2$ ($c=0.9$, $b=0.2$, and $k=0.5$)

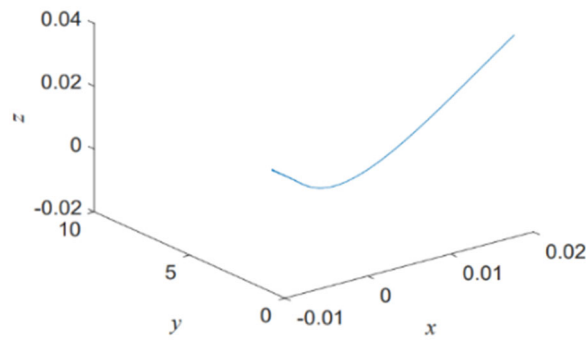


Figure 6. The phase diagram of Equation (2): (1) $(x-y)$, (2) $(x-z)$, (3) $(y-z)$ as $a=1.46$ ($c=0.9, b=0.2,$ and $k=0.5$)

Fixing the parameters $a=0.6, b=0.2, k=0.5$, we draw the bifurcation diagram of Equation (2) (Figure. 7). We find that when the parameter c is less than 2, the system is chaotic, and when the parameter c is greater than 2, the system exhibits a stable state. We draw the phase diagram of the system, taking the following parameters $c=1.8$ (Figure. 8), $c=2.1$ (Figure. 9), $c=2.0$ (Figure. 10). It can be observed that the system shows chaotic when the parameters $c=1.2, c=1.9$ are taken. When the parameter $c=2.0$ is taken, the system shows stable. The phase diagrams are consistent with theoretical derivation and bifurcation diagrams. We also verified these conclusions by calculating the values of the Lyapunov exponent.

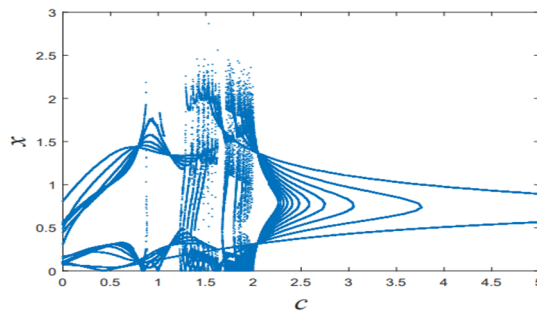


Figure 7. The phase diagram of Equation (2): (1) $(x-y)$, (2) $(x-z)$, (3) $(y-z)$ as c ($a=0.6, b=0.2,$ and $k=0.5$)

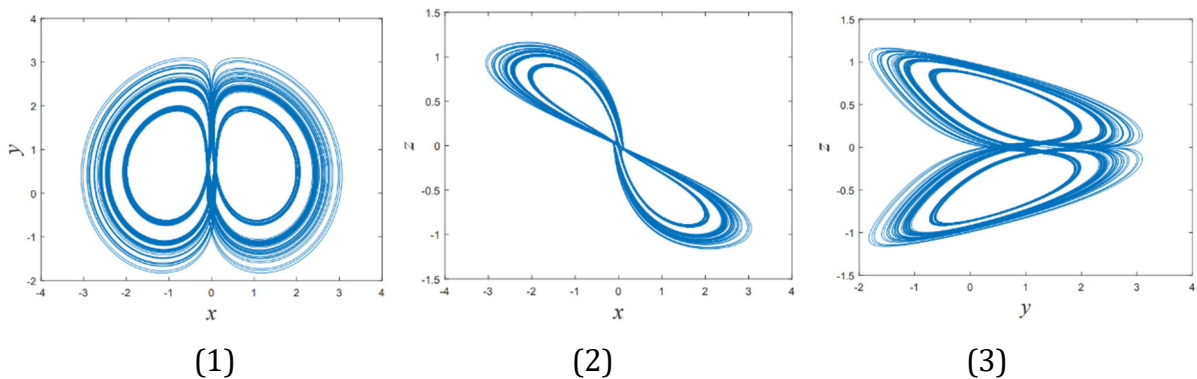


Figure 8. The phase diagram of Equation (2): (1) $(x-y)$, (2) $(x-z)$, (3) $(y-z)$ as $c=1.2$ ($a=0.6, b=0.2,$ and $k=0.5$)

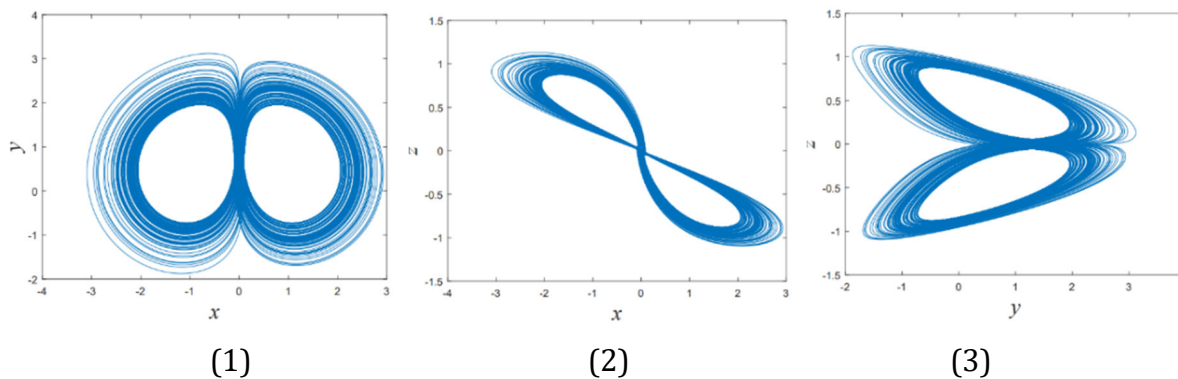


Figure 9. The phase diagram of Equation (2): (1) $(x-y)$, (2) $(x-z)$, (3) $(y-z)$ as $c=1.9$ ($a=0.6, b=0.2,$ and $k=0.5$)

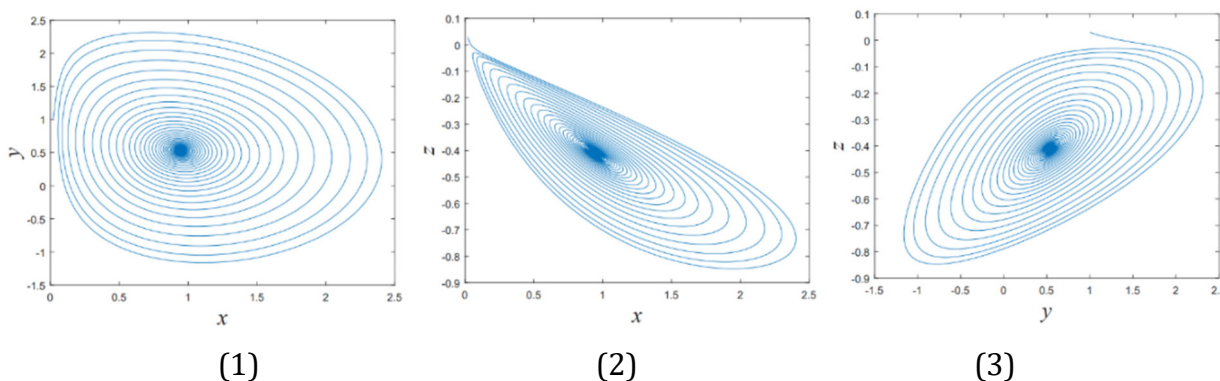


Figure 10. The phase diagram of Equation (2): (1) $(x-y)$, (2) $(x-z)$, (3) $(y-z)$ as $c=2.0$ ($a=0.6, b=0.2,$ and $k=0.5$)

4. CALCULATION OF LYAPUNOV EXPONENTS

Lyapunov exponent refers to the rate at which two adjacent orbits in phase space separate or close to each other exponentially. One of the most important features of chaotic systems is their sensitive dependence on initial values, and the Lyapunov exponent quantifies this sensitive dependence. When the Lyapunov exponent of a system reaches a positive value, the system is chaotic.

In this paper, the fourth order Runge Kutta method is used to discrete the financial chaotic system, and then the Lyapunov exponents are calculated according to the QR decomposition method of the discrete system. We consider the influence of parameters on the system and control the variates. We calculate the Lyapunov exponents of the system under different parameter values, and determine the stability of the system under different parameters.

Fixed parameters $b = 0.2, c = 0.9, k = 0.5$, changed parameters a , the maximum Lyapunov exponent of the system are 1.381 and 1.825 when $a=0.8$ and $a=1.2$ are taken. The Lyapunov exponents of the system are all greater than 0, indicating that the system exhibits a chaotic state under such parameter values. The maximum Lyapunov exponent of the system are -0.546 when $a=1.46$ is taken. The Lyapunov exponents of the system is less than 0, indicating that the system exhibits state.

Fixing the parameters $a=0.6, b=0.2, k=0.5$, changed parameters c , the maximum Lyapunov exponent of the system are 1.861 and 0.564 when $c=2$ and $c=2.3$ are taken. The Lyapunov exponents of the system are all greater than 0, indicating that the system exhibits a chaotic state

under such parameter values. The maximum Lyapunov exponent of the system are -0.324 when $c=2.0$ is taken. The Lyapunov exponents of the system is less than 0, indicating that the system exhibits state.

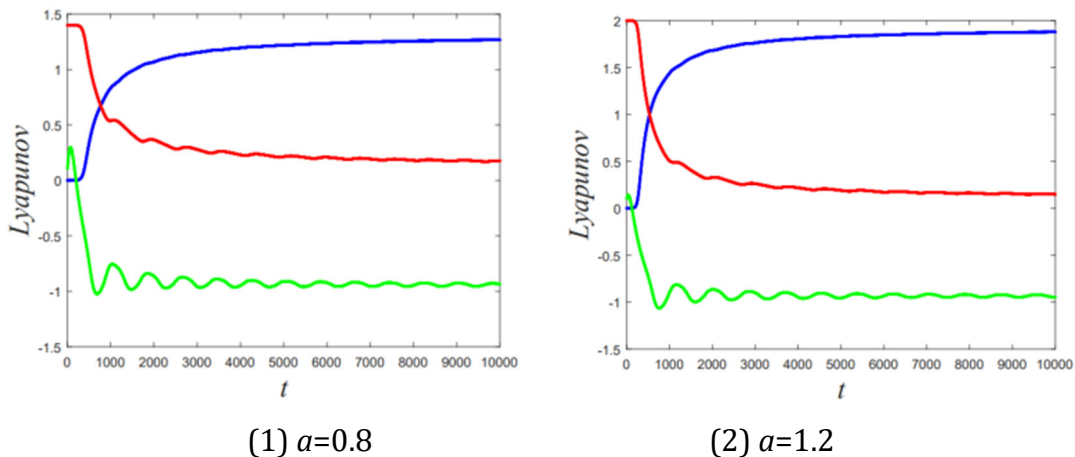


Figure 11. The Lyapunov exponent diagram of Equation (2), ($b=0.2, c=0.9$ and $k=0.5$)

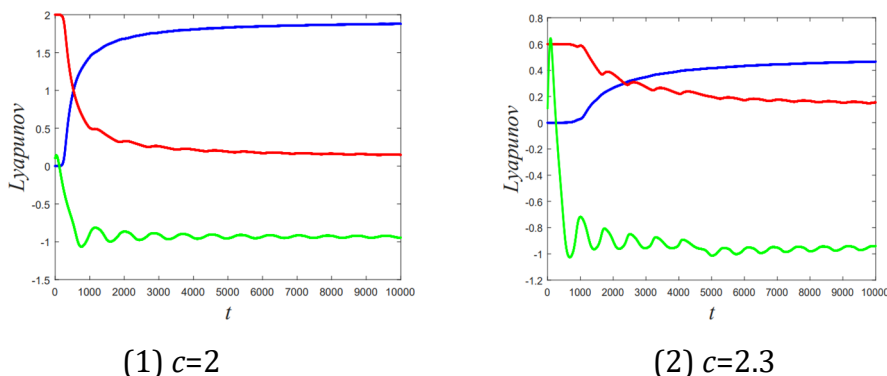


Figure 12. The Lyapunov exponent diagram of Equation (2), ($a=0.6, b=0.2,$ and $k=0.5$)

5. CONCLUSION

In this paper, the dynamics of financial chaotic system is analyzed. The central manifold function is obtained by using the central manifold theorem, and the Lyapunov exponent is calculated. Combined with phase diagrams, the system has a more comprehensive understanding. It is found that the selection of parameters in the system is very important. The smaller the savings or the greater the system fluctuation will lead to chaos; the parameter a should not be too large, which will easily lead to the lack of vitality of the economic market. Through the analysis of this research, it provides some ideas for decision-making institutions to regulate this kind of financial system, so as to make the economy relatively stable and avoid financial risks.

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REFERENCES

- [1] Liu Bingzheng. *Nonlinear Dynamics and Chaos Elements*. Changchun: Northeast Normal University Press, 1994.
- [2] Chen Yushu. *Bifurcation and Chaos Theory of Nonlinear Vibration Systems*. Beijing: Advance Education Press, 1993.
- [3] Lorenz E N. Deterministic Nonperiodic Flow. *J Atmospheric Sci.*, 1963, 20:130-141.
- [4] Henon M N. A Two Dimensional Mapping with a Strange Attractor. *Commun Math. Phys.*, 1976, 50 (1): 69-77.
- [5] Rosler E. An Equation for Continuous Chaos. *Phys Lett A*, 1976, 57:397-398.
- [6] Ruelle D, Takens F. On the Nature of Turbulence. *Commun Math Phys.*, 1971, 20(1):167-192.
- [7] May R M. Simple Mathematical Models with Very Complicated Dynamics. *Nature*, 1976, 261:459-467.
- [8] Feigenbaum M J. Quantitative Universality for a Class of Non-Linear Transformations. *J Statist Phys.*, 1978, 19: 25-52.
- [9] O E Rossler. An equation for continuous chaos. *Physics Letters A*, 1976, 57(5): 397-398.
- [10] L Chua, M Komuro and T Matsumoto. The double scroll family. *IEEE Transactions on Circuits and Systems*, 1986, 33(11):1072-1118.
- [11] Guo Feng, Ding Yin, and Li Jiacheng. Neimark-Sacker Bifurcation and Controlling Chaos in a Three-Species Food Chain Model through the OGY Method. *Discrete Dynamics in Nature and Society*, 2021, Article ID 6316235, <https://doi.org/10.1155/2021/6316235>
- [12] U Saeed, I Ali, and Q Din. Neimark–Sacker bifurcation and chaos control in discrete-time predator-prey model with parasites. *Nonlinear Dynamics*, 2021, 94: 2527-2536.
- [13] Melkikh, A V, R Y Beregov, and M I Sutormina. Strange attractors and nontrivial solutions in games with three players. *Chaos, Solitons & Fractals*, 2022, 154:111598.
- [14] Faqiang Wang, Hongbo Cao, Dingding Zhai. A New 4D Piecewise Linear Multi-scroll Chaotic System with Multistability and Its FPGA-Based Implementation. *Complexity*, 2021:1-15.
- [15] Sambas A, Vaidyanathan S, Sukono, et al. Multistability in a New Chaotic System with Biscuit-Shaped Equilibrium, its Analysis, Synchronization and Circuit Design. *An interdisciplinary journal of discontinuity, nonlinearity, and complexity*, 2022, (3):11.
- [16] D S Huang and H Q Li. *Theory and Method of the Nonlinear Economics*. Sichuan University Press Co., Ltd., Chengdu, China, 1993.
- [17] J Móczár. Alfred Cowles' Investments: New Methods in Econometrics and Mathematical Economics. 2021: 382-388.
- [18] Gardini L, Lamantia F, Radi D, et al. Nonlinear dynamics in economic modelling. *Decisions in Economics and Finance*, 2021, 44: 1213-1224.
- [19] Vaidyanathan S, Sambas A, Mujiarto, et al. A New 4-D Multistable Hyperchaotic Two-Scroll System, its Bifurcation Analysis, Synchronization and Circuit Simulation. *Journal of Physics Conference Series*, 2021, 1764(1): 012206.
- [20] Gularte K, Gomez J, Vargas J, et al. Underactuated Synchronization Scheme of a Hyperchaotic Finance System, 14th IEEE International Conference on Industry Applications (INDUSCON). IEEE, 2021.
- [21] Singh S, Mathpal S, Azar A T, et al. Multi-switching synchronization of nonlinear hyperchaotic systems via backstepping control. *Backstepping Control of Nonlinear Dynamical Systems*, 2021.

- [22] J H Ma and Y S Chen. Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (I), *Applied Mathematics and Mechanics*, 2001, 22(11): 1240-1251.
- [23] J H Ma and Y S Chen. Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (II), *Applied Mathematics and Mechanics*, 2001, 22(12): 1375-1382.
- [24] Wang B, Jahanshahi H, Bekiros S, Chu Y M, Alsaadi F E, & Alassafi M O, et al. Tracking control and stabilization of a fractional financial risk system using novel active finite-time fault-tolerant controls. *Fractals: An interdisciplinary journal on the complex geometry of nature*, 2021, 6(29):18-22.
- [25] Pechuk V D, Krasnopolskaya T S, Pechuk E D. Maximum Lyapunov Exponent Calculation. *Chaotic Modeling and Simulation International Conference*. Springer, Cham, 2022.